THE BENJAMIN-ONO AND BBM EQUATIONS AFTER REGULARIZATION

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Abstract

The periodic solutions of the Benjamin-Bona-Mahony (BBM) equation and the regularized Benjamin-Ono (rBO) equation demonstrate a characteristic of nonlinear stability under wavelength-sharing perturbations is the subject of this work. This work clarify that these perturbations are stable through analytical analysis, which advances our knowledge of the stability characteristics of these nonlinear wave equations. Further, in periodic and non-periodic (line) situations, this work enhances the global well-posedness conjecture associated with the rBO equation. The difficulties raised in these settings by the Cauchy problem are emphasized a great deal. In particular, we show that the widely used iteration strategy The application of the Duhamel formula to the Cauchy problem associated with the rBO equation proves to be ineffective in resolving such issues with negative Sobolev indices. This result points to a significant flaw in the usual iterative methods for solving these equations, requiring the development of new techniques or frameworks to handle the Cauchy issue when such indices are present. Additionally, the research advances our knowledge of the mathematical frameworks that underlie nonlinear wave equations, particularly with regard to the effects of regularization on stability and well-posedness. The findings have ramifications for theoretical studies as well as real-world applications that include wave phenomena' stability and solvability. In summary, this study offers fresh perspectives on the behavior of the rBO and BBM equations and presents a more thorough framework for examining their stability and resolving related Cauchy issues.

Keywords: Periodic wave solutions, Nonlinear wave stability, Modified Benjamino-Ono equation BBM equation (Benjamin-Bona-Mahony)

1. Introduction

The primary objective that underpins the current research initiative is to undertake a comprehensive and meticulous examination of the stability attributes and characteristics that are intrinsically linked to the periodic traveling wave solutions, which are of paramount importance in elucidating a myriad of phenomena, particularly within the context of two prominent nonlinear wave models that are widely employed to depict the complex dynamics associated with fluid flow; these models include, most notably, the regularized Benjamin-Ono equation, (ABD ELRAHMANet al. 2019) which is employed in this discourse will be designated as the rBO equation, in addition to the Benjamin-Bona-Mahony equation, which is frequently referred to by its acronym, the BBM equation. It is of considerable significance to highlight that the rBO equation is precisely articulated through a formal definition that encapsulates the fundamental components and mathematical constructs that govern the intricate behavior of the wave solutions we are investigating, (AJINKYAet al. 2020) thereby providing a robust framework for understanding the underlying mechanisms at play. Furthermore, this research seeks to contribute to the broader academic dialogue surrounding nonlinear wave phenomena by elucidating the stability characteristics inherent in these equations, thus enriching our understanding of their applications and implications in the field of fluid dynamics.

$$u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0 \tag{1}$$

where H represents the Hilbert transform, which can be characterized through the Fourier transform in the following manner:

$$sgn(k) = \{-11 if\ k < 0 if\ k > 0 \quad (2)$$

The regularized iteration of the Benjamin-Ono equation functions as a sophisticated mathematical construct that proficiently delineates the temporal progression and developmental characteristics of long-crested waves, (AL-ALAWYet al. 2018) which are intricate phenomena emerging at the boundary where two immiscible fluids converge and engage in complex interactions. (ANSARI et al. 2019) This specific equation is of paramount importance across a diverse array of practical scenarios, which encompass, although are not confined to, the pycnocline that is situated within the profound depths of the ocean, as well as the two-layer hydrodynamic system that is generated as a consequence of the influx of freshwater from rivers into marine ecosystems, a subject that has been meticulously explored in the scholarly contributions of (BAABU et al. 2022). It is crucial to underscore the fact that this regularized equation bears a formal equivalence to the original Benjamin-Ono equation, which is frequently abbreviated as the BO equation for the sake of convenience and clarity in reference and analysis throughout the vast expanse of scientificliterature.

$$v_t + v_x + vv_x - \mathcal{H}v_{xx} = 0 \quad (3)$$

(BHATTACHARYAet al. 2011) Initially introduced by the highly regarded mathematician Benjamin in his influential and foundational work, and subsequently elaborated upon by Ono in his extensive and thorough examination, this specific model equation fundamentally engages with the same theoretical structure as the rapidly oscillating Boussinesq equation, which is commonly denoted in scholarly literature as rBO. More precisely, (HERNÁNDEZ-HERNÁNDEZ et al. 2020) when one considers appropriately defined and suitably constrained initial conditions, the solutions represented by uuu in the context of equation (4) and vvv in relation to equation (1.2) display an astonishingly close resemblance to each other when they are evaluated for time ttt within the specified interval of [0, T][0, T][0, T], where it is imperative that TTT is sufficiently extensive to ensure the rigorous validity of the analysis being conducted. For additional insights and a more profound exploration into this intricate and complex subject matter, Moreover, the aforementioned reference offers an exhaustive comparative analysis that serves to illuminate the respective benefits and drawbacks associated with the utilization of equation (5) as opposed to equation (6) for the effective modeling of the sophisticated dynamics that it is essential to meticulously regulate and manage the propagation of long waves while ensuring that the amplitude remains at a modest and controlled level, thereby facilitating a balanced approach to the intricacies involved in wave dynamics.

(HUSSAINet al. 2023) This scholarly research endeavor, which places significant emphasis on the myriad of challenges that are inherently linked to the exploration and understanding of traveling wave solutions, seeks to systematically investigate and illuminate a crucial qualitative aspect pertaining to the complex nature of nonlinear dispersive equations that are prevalent in various fields of study. (KUMAR et al. 2021) The outcomes and interpretations of this investigation may vary considerably based on the specific boundary conditions that are imposed upon the mathematical framework and physical scenarios under consideration, thereby influencing the overall dynamics and characteristics of the solutions derived from these equations. In light of this, it becomes essential to meticulously analyze the interplay between the traveling wave solutions and the imposed boundary conditions, as this relationship is pivotal for advancing our comprehension of the intricate behaviors exhibited by nonlinear dispersive equations in diverse applications that exert a significant influence over the morphological characteristics of the wave, these solutions can manifest either as solitary waves, which are characterized by their localized nature, or they can take the form of periodic waves, (OGBEZODEeet al. 2023) which exhibit a repetitive structure over time. Over the course of over the course of the preceding two decades, an extensive and substantial corpus of academic literature has been dedicated to the rigorous examination and thorough investigation of the existence of solitary-wave solutions, their nonlinear robustness, and potential instability, which are of considerable interest within the field of mathematical physics. This expansive and multifaceted research endeavor has employed a diverse array of sophisticated methodologies and theoretical frameworks, (RUKHSAR et al. 2022) which have collectively played a crucial role in enhancing and enriching the scholarly discourse surrounding this complex topic. A plethora of innovative and avant-garde techniques have been developed and applied in the pursuit of a deeper understanding of these phenomena, reflecting the dynamic and evolving nature of research in this area. Consequently, the culmination of these scholarly efforts has significantly advanced the field, providing valuable insights and fostering further inquiry into the underlying principles governing solitary waves and their behaviors meticulously formulated to accurately identify these solitary responses, and sufficiently comprehensive parameters have been meticulously delineated to ascertain their stability or instability, as has been thoroughly detailed in a wide range of scholarly sources, (SHEHA et al. 2023) In stark contrast to this extensive body of work, the exploration of periodic traveling wave solutions has not garnered as much scholarly attention; however, it is noteworthy that In the last few years, there has been a remarkable and significant increase in the volume and intensity of scholarly research endeavors and investigative activities across various fields of study, (TANVIRet al. 2023) which highlights a growing trend towards a more rigorous and comprehensive exploration of diverse topics that reflect the evolving interests and needs of the academic community and society at large related to this particular area of study, with a plethora of relevant papers emerging.

The resolutions concerning recurring traveling waves of interest are given by $\psi(x,t) = \phi(x - ct)$, When the differentiable function ϕ:ℝ→ℝ is with period 2L with c≠1. Executing the integration process, assigning the integration constant a value of zero, and subsequently substituting this waveform into equation (4) gives:

$$\phi_c(x) = \frac{4(c-1)}{1 + \left(\frac{c-1}{c}x\right)^2} \tag{4}$$

By employing Employing Benjamin's characterization of The resolution for oscillatory propagating waves pertaining to the BO problem, we establish the existence of a smooth curve $c \to \phi_c$possesses a foundational interval characterized by Equation (5) 2L has positive, even, periodic solutions.

$$\phi_c(\xi) = \frac{2c\pi}{L} \frac{\sinh(\eta)}{\cosh(\eta) - \cos\left(\frac{\pi\xi}{L}\right)} \tag{5}$$

with η satisfying

$$\eta(c) = \tanh^{-1}\left(\frac{c\pi}{(c-1)L}\right) \tag{6}$$

with L>πL > \piL>π and c>LL−πc > \frac{L}{L - \pi}c>L−πL. The primary objective of our discourse in this context is to engage in a thorough and comprehensive investigation into the stability characteristics of these periodic waveforms, which are of critical importance in the broader field of wave dynamics. In light of our extensive and well-established expertise concerning the intricate nature of nonlinear dispersive evolution equations, we have come to a profound understanding that the presence of traveling waves, when they occur, is instrumental and pivotal in shaping the evolution of a wide array of disturbances that may arise in various systems. Consequently, it is with great expectation and anticipation that we assert that the detailed examination of the stability of periodic waves will emerge as a significant domain of inquiry, rich with complexity and ripe for exploration, offering numerous insights into the underlying phenomena. Therefore, it is our firm belief that delving into this area will not only enhance our theoretical understanding but also contribute valuable knowledge to the scientific community at large, highlighting the intricate interplay between wave stability and nonlinear dynamics.

Of constructive, even, periodic solutions characterized by a fundamental period for Equation (7)

$$E(u) = \frac{1}{2}\int \left(u\mathcal{H}u_x - \frac{1}{3}u^3\right)dx \quad \text{and} \quad F(u) = \frac{1}{2}\int (u^2 + u\mathcal{H}u_x)dx \tag{7}$$

and that $\phi_c$constitutes a significant element of the operational $E + (c - 1)F$, In other terms, we employ the following equation given in (8). Additionally, it is necessary to consider the a specific spectral configuration associated with the nonlocal operator.

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$$\mathcal{L} = c\mathcal{H}\,\partial_x + c - 1 - \phi_c \quad (8)$$

in the domain of periodic structures, particularly when scrutinizing the operator referred to as LLL, it is of particular significance to note that this operator is endowed with a singular negative eigenvalue, which is remarkable not only for its existence but also for its simplicity, while simultaneously, the eigenvalue corresponding to zero also demonstrates a similar simplicity and is associated with the eigenfunction denoted as ϕc′. Additionally, it is imperative to underscore the fact that the remaining portion of the spectrum is distinctly delineated commencing from zero which consequently amplifies our comprehension of the spectral characteristics that are at play. To meticulously establish these spectral conditions in a rigorous manner, we draw upon the groundbreaking theory, which adeptly leverages the beneficial attributes intrinsic to the Fourier transform of the eigenfunction ϕc′. In order to proficiently derive the Fourier coefficients that pertain to the function ϕc′, we employ the Poisson Summation theorem, a fundamental and widely recognized tool that plays a crucial role in constructing the profile as delineated in equation (8). This specific approach to determining periodic profiles not only reveals substantial potential but also represents a promising pathway for the investigation of analogous issues within this particular field of inquiry. Ultimately, this thorough investigation significantly enriches the overarching dialogue surrounding spectral analysis and its myriad applications across diverse mathematical frameworks, thereby contributing to the ongoing evolution of knowledge in this area. Moreover, the implications of this research extend beyond theoretical boundaries, suggesting possible practical applications that could emerge from a deeper understanding of these spectral properties. Consequently, it is evident that the pursuit of this line of inquiry holds the potential to yield fruitful insights and advancements in both theoretical and applied mathematics.

Previous scholarly investigations that were meticulously conducted by esteemed researchers Spector and Miloh, as duly cited in their academic publication, have significantly clarified and illuminated the findings Given that the Benjamin-Ono (BO) equation is linked to a specific normalized category of periodic solutions, which is characterized by the specific profiles that are delineated in the mathematical formalism of equation (9), demonstrate a notable form of linear stability that is essential for understanding the dynamics of such solutions. The remarkable and noteworthy outcomes of their comprehensive research endeavors were achieved through the strategic leveraging of the inherent property that the BO equation possesses, which is its classification as completely integrable, thereby facilitating the effective application of the inverse scattering transform, a highly potent and sophisticated analytical tool in the field of mathematical physics. However, it is paramount to highlight that the current study distinctly diverges from the methodological paradigm established by their previous work, as it consciously opts not to employ the particular technique that they utilized for the thorough and exhaustive analysis of the operator LLL as articulated in equation (9). As a result of this deliberate methodological shift, this indicates a clearly distinct approach in the examination and analysis of the operator, which may potentially yield a variety of different insights, interpretations, or conclusions that stand in contrast to those that were derived from the previously established and utilized technique employed by Spector and Miloh. Such divergence in methodological framework may not only enrich the discourse within this domain of study but also pave the way for new avenues of research that could further enhance our understanding of the complex dynamics associated with the BO equation and its related operators. Ultimately, this study aspires to contribute to the ongoing academic dialogue by providing fresh perspectives and possibly innovative findings that could challenge or expand upon the established knowledge in the field.

We extend the theoretical framework to a more extensive category of regularized equations in the latter portion of this study. We investigate a diverse array of equations of the form

$$u_t + u_x + u^p u_x + Hu_t = 0 \quad (9)$$

in the realm of periodic functions, the symbol H signifies a differential or pseudo-differential operator, with p being an integer such that p≥1. Many practical equations exhibit this form. For example, with $H = -\partial_x^2$, The generalized Benjamin-Bona-Mahony equation has been derived. Notably, this simplifies to the BBM equation when p is equal to 1 with $H = H\,\partial_x$, we derive the generalized regularized Benjamin-Ono equatiom. Such generalizations previously investigated within the context of solitary wave phenomena and analyzed the spectral stability of periodic traveling wave solutions pertaining to the generalized BBM equation in a periodic framework. For c>1, she established the criterion for spectral stability when $1 \leq p \leq 2$and establishing the pivotal velocity c_p, for p≥3, at which the wave phenomena exhibit stability for $c \in \left(c_p, \frac{p}{p-3}\right)$ and unstable for $\in \left(1, c_p\right) \cup \left(\frac{p}{p-3}, \infty\right)$. Additionally, A specific collection of BBM equations was analyzed to explore the orbital stability of various generalized BBM and Camassa-Holm equations defined by this form.

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} = 0, \quad \omega \in \mathbb{R} \tag{10}$$

They demonstrated the existence of solutions characterized by the cnoidal form has been established, yet the verification of the orbital stability of these solutions has only been demonstrated in the specific scenario where ω=0. In relation to the periodic wave solutions pertinent to the equations under investigation, we meticulously articulate appropriate criteria that facilitate the attainment of nonlinear stability form (11), which exhibit a periodic structure analogous to that of the foundational wave, thereby ensuring that the system can withstand any form of periodic perturbation that may arise. To illustrate the practical application of our theoretical findings, we provide a detailed example wherein we rigorously analyze The resolutions of the Benjamin-Bona-Mahony equation characterized by cnoidal waveforms, commonly referred to as the BBM equation, which is characterized by a specific wave profile that serves as a focal point of our study and highlights the nuances of stability in nonlinear wave dynamics. This comprehensive exploration not only contributes to the existing body of knowledge surrounding wave stability but also emphasizes the critical interplay between mathematical formulations and the physical phenomena they aim to describe, thereby enhancing our understanding of nonlinear wave behavior in various contexts. By

$$\phi_c(x) = \alpha_1 + \alpha_2 \mathrm{cn}^2(\alpha_3 x; k) \tag{11}$$

It is our firmly held belief that the extensive and thorough stability analysis we have meticulously conducted yields a plethora of profound insights that are directly relevant to the complex and multifaceted behavior exhibited by dispersive systems, which are not only fundamental to a wide array of physical phenomena but also play a crucial role in various practical applications across multiple scientific fields. Furthermore, the innovative methodology we have adeptly employed in our comprehensive analysis has the remarkable potential to significantly enhance and streamline practical numerical simulations; this is particularly pertinent for those simulations that are aimed at accurately modeling intricate scenarios such as the dynamic interactions of water waves occurring at the interface between two distinctly different fluids or the intricate behavior of gravity waves within the long-wave regime, a phenomenon that is essential for a deeper understanding of numerous natural processes and events. In light of these carefully considered points, our research not only makes substantial contributions to the theoretical understanding of dispersive systems but also paves the way for the development of more accurate and efficient computational techniques, which can be effectively applied across various disciplines within the expansive field of fluid dynamics and potentially beyond its immediate boundaries. Thus, the implications of our findings extend far beyond mere theoretical discourse, impacting practical applications and fostering advancements in computational methods that hold promise for a myriad of scientific inquiries.

A quartet of parameterized spatially periodic traveling wave solutions pertaining to the generalized Benjamin-Bona-Mahony equation, denoted by the symbol φ(·;p), has recently undergone a stability. Specifically, Johnson demonstrated that periodic waves with For all periodic disturbances occurring within the continuous submanifold, waves characterized by a sufficiently large fundamental period or sufficiently extended wavelengths exhibit nonlinear stability for the range of 1≥p<4 defined by$H^1_{per}$of codimension two,

$$\Sigma_p = \Big\{ f \in H^1_{per}: \int f(x)dx = \int \varphi(x;p)dx, \int f^2(x) + f'^2(x)dx = \int \varphi^2(x;p) + \varphi'^2(x;p)dx \Big\} \quad (12)$$

We intend to implement our methodology to establish a stability framework for both the revised BBM equation (p=2) and the notable BBM equation (p=4)under any periodic perturbation.

This study's subsequent focus pertains to the well-posedness dilemma associated with In the context of periodic Sobolev spaces, the rBO equation$H^s_{\text{per}}([-L, L])$ or $H^s(\mathbb{R})$. Indeed, we illustrate that when s>1/2, the rBO is consistently well-posed. To our knowledge, this information has not been previously documented. Our results build upon Mammeri's estimation from regarding the mathematical formulation for periodic rBO is delineated. In his scholarly work, he articulates a comprehensive theorem regarding the equation's global well-posedness.

$$u_t + u_x + \alpha u u_x + \beta \mathcal{H} u_{xt} = 0 \quad (13)$$

where α and β are constants such that $0 < \alpha, \beta \leqslant 1$. Mammeri also proved that the Cauchy problem associated to Eq. (1.11) is globally well-posed in $H^s_0([-L, L])$, for $s > 1/2$, where $H^s_0([-L, L])$ means the elements f of $H^s([-L, L])$has zero mean. Given our aspiration to provide a stable environment,

The rBO equation in $H^s(\mathbb{R})$ with $s \geq \frac{3}{2}$. Given the conservation laws in (1.7), $H^{1/2}(\mathbb{R})$ (or $H^{1/2}_{\text{per}}$) appears to be a suitable space for studying the Cauchy problem for Eq. (13). This problem remains open in $H^s(\mathbb{R})$ (or H^s_{per}) with $s \leq \frac{1}{2}$, and one of the objectives of this paper is to identify some challenges in solving it through iterative methods. Specifically, demonstrate that the data-to-solution map cannot be C^2 for $s < 0$ in both the periodic and non-periodic case.

Ultimately, the meticulous structure and precise organization of this scholarly article are clearly delineated and articulated as follows: embark on a comprehensive introduction of a meticulously crafted series of notational conventions that will serve as a consistent and reliable framework for reference as well as clarity throughout the entirety of this expansive and in-depth work. Transitioning undertake a rigorous and thorough examination, coupled with a subsequent proof of both global well-posedness and ill-posedness results, addressing these complex phenomena within the multifaceted contexts of both periodic and nonperiodic settings, thereby illuminating the intricate complexities that are inherently present within these advanced mathematical constructs. In Section 4, delve deeply into an exploration of the existence of periodic traveling waves, employing the powerful and widely respected analytical tool known as the Poisson Summation theorem to robustly substantiate our claims and findings regarding such waves, thus enhancing the overall depth of our investigation. The discussion then seamlessly proceeds to Section 5, where we meticulously outline the crucial spectral properties that are absolutely indispensable for establishing the nonlinear stability of the systems under consideration, thereby laying a solid groundwork for our subsequent and more detailed analyses. Advancing to Employing the foundational principles, we meticulously assess the stability it examine periodic traveling waves and concepts that have been previously articulated in the literature, thereby significantly enhancing our understanding of the dynamic behavior of waves. Finally, present a comprehensive and expansive extension of the theoretical framework that is pertinent to the regularized Benjamin-Ono equation (rBO), utilizing this broadened theoretical approach to rigorously demonstrate the stability of cnoidal waves that are intricately associated with the well-known and widely studied Benjamin-Bona-Mahony (BBM) equation. Through this systematic and well-structured organization, we aim to provide both clarity and coherence in the presentation of our research findings, ensuring that readers can easily navigate through the complexities and nuances of the subject matter. Each section is thoughtfully designed not only to build upon the knowledge established in the previous sections but also to contribute significantly to a holistic understanding of the phenomena being meticulously studied. Thus, we earnestly invite readers to engage deeply and critically with the rich material presented within this article, fostering a greater appreciation for the intricate relationships that exist within the expansive realm of wave dynamics and stability analysis.

2. Notes and introduction

Our notation follows standard conventions in partial differential equations; for more details, Let $P = C^{\infty}_{\text{per}}$ denote the set of all functions $f: \mathbb{R} \to \mathbb{C}$ that are C^{∞} and periodic with period $2L > 0$. Let P' represent the set of periodic distributions. If $\Psi \in P'$, we denote the value of Ψ at φ by $\Psi(\varphi) = \langle \Psi, \varphi \rangle$. The Fourier transform of $\Psi$is the function $\hat{\Psi}: \mathbb{Z} \to \mathbb{C}$ given by the formula $\hat{\Psi}(k) = \frac{1}{2L}\langle \Psi, \Theta_k \rangle$, where $\Theta_k(x) = \exp\left(\frac{\pi i k x}{L}\right)$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Therefore, if Ψ is a periodic function with period $2L$, we have

$$\widehat{\Psi}(k) = \frac{1}{2L}\int_{-L}^{L} \Psi(x) e^{-\frac{ik\pi x}{L}} dx \tag{14}$$

For $s \in \mathbb{R}$, the Sobolev space of order s, denoted by $H^s_{\text{per}}([-L, L])$, consists of all $f \in P'$ such that $(1 + |k|^2)^{s/2}\hat{f}(k) \in \ell^2(\mathbb{Z})$. The norm is given by $\| f \|^2_{H^s_{\text{per}}} = \frac{1}{2L}\sum_{k=-\infty}^{\infty}(1 + |k|^2)^s |\hat{f}(k)|^2$. For $s = 0, H^0_{\text{per}}$ is denoted by L^2_{per}, with $(f, g) = \int_{-L}^{L} \bar{f} g dx$ and $\|\cdot\|_{H^0_{\text{per}}} = \|\cdot\|_{L^2_{\text{per}}}$.

If Y is a Banach space and $T > 0$, then $C([0, T]; Y)$ is the space of continuous functions from $[0, T]$ to Y. For $k \geq 0, C^k([0, T]; Y)$ denotes the subspace of functions $t \mapsto u(t)$ such that $\partial^j_t u \in C([0, T]; Y)$ for $0 \leq j \leq k$, where the derivative is understood in the sense of vectorvalued distributions. This space is equipped with the standard norm

$$\| u \|_{C^k([0,T];Y)} = \sum_{j=0}^{k} \max_{0 \leqslant t \leqslant T} \left\|\partial^j_t u(t)\right\|_Y \tag{15}$$

Finally $\mu(A)$denotes the Lebesgue measure associated with the set A.

Subsequently, the Poisson Summation theorem is established. The identification of periodic traveling wave solutions for the rBO and BBM equations, respectively, will prove advantageous in equation 16 and 17.

Theorem Let $\hat{f}^{\mathbb{R}}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ and $f(x) = \int_{-\infty}^{\infty} \hat{f}^{\mathbb{R}}(\xi) e^{2\pi i x \xi} d\xi$ satisfy

$$|f(x)| \leqslant \frac{A}{(1+|x|)^{1+\delta}} \text{ and } \left|\hat{f}^{\mathbb{R}}(\xi)\right| \leqslant \frac{A}{(1+|\xi|)^{1+\delta}} \tag{16}$$

where $A > 0$ and $\delta > 0$ (then f and $\hat{f}$ can be assumed continuous functions). Thus, for $L > 0$

$$\sum_{n=-\infty}^{\infty} f(x + 2Ln) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} \hat{f}^{\mathbb{R}}\left(\frac{n}{2L}\right) e^{\frac{\pi i n x}{L}} \tag{17}$$

3. Outcomes concerning the rBO regarding criteria of well-posedness and ill-posedness.

In the preliminary stages of our extensive and meticulous inquiry into the ramifications of the rBO equation, we will systematically and rigorously establish a comprehensive array of pivotal findings that are fundamentally intertwined with the well-posedness issue, which, as we will elucidate, can emerge in both periodic and nonperiodic scenarios, consequently offering a solid and coherent framework for our analytical endeavors. It is of paramount significance to emphasize that the theoretical insights gleaned from the periodic context will undeniably serve as an essential foundation as we embark upon the complex exploration of the nonlinear stability associated with the waveform solutions that are derived, ultimately empowering us to formulate deeper and more nuanced interpretations regarding their dynamic behavior under a multitude of varying conditions and circumstances.

In the context of a Banach space (X), we characterize the initial value problem (IVP) associated as locally well-posed if there exists a unique solution within a specified temporal interval $[-T, T]$The resolution delineates a continuous trajectory in the space X within the specified interval (distinct existence)$[-T, T]$ whenever the initial data belongs to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e., we have the continuity of the application $u_0 \to u(t)$ from X to $C([0,T];X)$. We say that the IVP associated to and is globally well-posed in X if the same properties hold for all time $T > 0$. If some property in the definition of locally well-posed fails, we say that the IVP is ill-posed.

3.1. Ill-posedness in H^s_{per} and $H^s(\mathbb{R})$ with $s < 0$

In this subsection, we demonstrate that the data-to-solution mapping for the Cauchy problem associated with the rBO equation is not C^2 at the origin when considering initial data inH^S_{per} (or $H^s(\mathbb{R})$), with $s < 0$. As a result, the Contraction Principle cannot be applied to solve the integral equation (18), as detailed below.

To begin, we examine the problem in the periodic setting. For simplicity, we focus on functions of period 2π. We understand that the linearized problem associated with equation (18) and initial data ψ\psiψ yields the solution nnn.

$$u(x,t) = S(t)\psi(x) = \sum_{n=-\infty}^{+\infty} e^{inx-\frac{in}{1+|n|}t}\hat{\psi}(n) \tag{18}$$

Now, if u is solution of (19), then by Duhamel principle we have that

$$u(x,t) = S(t)\psi(x) - \int_0^t S(t-\tau)\Lambda[u(x,\tau)u_x(x,\tau)]d\tau \tag{19}$$

where $\widehat{\Lambda u}(n) = (1+|n|)^{-1}\hat{u}(n)$.

The following theorem is the principal result of this section.

Theorem. Let $s < 0$ and T a positive number. Then there does not exist a space X_T continuously embedded in $C([-T,T];H^s_{\text{per}})$ such that there exist $c_0 > 0$ satisfying

$$\| S(t)\psi \|_{X_T} \leqslant c_0 \| \psi \|_{H^s_{per}}, \quad \forall \psi \in H^s_{per} \tag{20}$$

and

$$\left\| \int_0^t S(t-\tau)\Lambda[u_x(\tau)u(\tau)]d\tau \right\|_{X_T} \leqslant c_0 \| u \|^2_{X_T}, \quad \forall u \in X_T \tag{21}$$

Proof. Suppose by contradiction that there exists such a space. Consider $\psi \in H^s_{\text{per}}$ and define $:= S(t)\psi$. Then, from (3.3) we have that $u \in X_T$ and since $X_T \hookrightarrow C([-T,T];H^s_{per})$, we get from (22) that

$$\left\| \int_0^t S(t-\tau)\Lambda[S(t)\psi(S(t)\psi)_x]d\tau \right\|_{H^s_{per}} \leqslant c_0 \| \psi \|^2_{H^s_{per}} \tag{22}$$

Next we prove that choosing ψ, appropriately, (23) does not hold. In fact, consider$(x) := N^{-s}\cos(Nx)$, with $N \in \mathbb{N}, N \gg 1$.

ψIt easy to see that $S(t)\psi(x) = N^{-S}\cos\left(Nx - \frac{N}{1+N}t\right)$. Then,

$$\varphi(x,t) := \int_0^t S(t-\tau)\Lambda[S(t)\psi(x)(S(t)\psi(x))_x]d\tau$$
$$= -\frac{1}{2}N^{-2s+1}\int_0^t S(t-\tau)\Lambda\left[\sin\left(2Nx - \frac{2N}{1+N}\tau\right)\right]d\tau \quad (23)$$

Now, using the specific form of Λ we obtain that

$$\int_0^t S(t-\tau)\Lambda\left[\sin\left(2Nx-\frac{2N}{1+N}\tau\right)\right]d\tau = -\frac{1}{2(1+2N)\gamma_N}\left[e^{i\left(2Nx-\frac{2N}{1+2N}t\right)} - e^{i\left(2Nx-\frac{2N}{1+N}t\right)}\right] + \frac{1}{2(1+2N)\gamma_N}\left[e^{-i\left(2Nx-\frac{2N}{1+N}t\right)} - e^{-i\left(2Nx-\frac{2N}{1+2N}t\right)}\right] \quad (24)$$

where $\gamma_N = \frac{2N^2}{(1+N)(1+2N)}$. Therefore

$$\varphi(x,t) = \frac{1}{2}N^{-2s+1}\frac{1}{\gamma_N(1+2N)}\left[\cos\left(2Nx-\frac{2N}{1+2N}t\right) - \cos\left(2Nx-\frac{2N}{1+N}t\right)\right] \quad (25)$$

Hence,

$$\| \varphi(\cdot,t) \|^2_{H^s_{per}} \sim N^{-4s}\left|e^{-i\frac{2N}{1+2N}t} - e^{-i\frac{2N}{1+N}t}\right|^2 (1+4N^2)^s \sim N^{-2s}\big(1-\cos(\gamma_N t)\big). \quad (26)$$

Note that $\| \psi \|^2_{H^s_{per}} \sim 1$, then for all $t \in (0,T)$ we have

$$\frac{\|\varphi(\cdot,t)\|_{H^s_{per}}}{\|\psi\|^s_{H^s_{per}}} \sim N^{-s}\big(1-\cos(\gamma_N t)\big)^{\frac{1}{2}} \quad (27)$$

Without loss of generality we can suppose $0 < T < 2\pi$. For $s < 0$ fixed, we obtain that

$$\frac{\|\varphi(\cdot,t)\|_{H^s_{per}}}{\|\psi\|^s_{H^s_{per}}} \longrightarrow +\infty \quad (28)$$

as $N \to +\infty$, for all $0 < t < T$, which contradict (29).

As a consequence we get the next result.

Corollary Fix $s < 0$. There does not exist a $T > 0$ such that admits a unique local solution defined on the interval $[-T,T]$ and such that for any fixed $t \in [-T,T]$ the map$\psi \longmapsto u(t)$

is C^2 differentiable at zero from H^s_{per} to H^s_{per} .

Proof. Consider the Cauchy problem

$$\begin{cases} u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0 \\ u(x,0) = \psi_\gamma(x), \quad 0 < \gamma \ll 1 \end{cases} \quad (29)$$

where $\psi_\gamma(x) := \gamma\psi(x)$. Suppose that $u(\gamma,t,x)$ is a local solution of (30) and the map data-solution is C^2 at the origin from H^s_{per} to H^s_{per}. Then

$$\frac{\partial u}{\partial \gamma}(\gamma,t,x)\Big|_{\gamma=0} = S(t)\psi(x) \quad (30)$$

and

$$\frac{\partial^2 u}{\partial \gamma}(\gamma,t,x)\Big|_{\gamma=0} = -2\int_0^t S(t-\tau)\Lambda[(S(\tau)\psi)(S(\tau)\psi)_x]d\tau \tag{31}$$

Using the assumption, we have

$$\left\|\int_0^t S(t-\tau)\Lambda[(S(\tau)\psi)(S(\tau)\psi)_x]d\tau\right\|_{H^s_{per}} \leqslant c_0 \parallel \psi \parallel^2_{H^s_{per}} \tag{32}$$

The final estimate mirrors that in equation (33), which was demonstrated to be invalid in the preceding theorem. We now extend the same type of results to the nonperiodic setting. In this context, we have

$$S(t)\psi(x) = \int_{\mathbb{R}} \hat{\psi}(\xi)e^{i\left(\xi x-\frac{\xi}{1+|\xi\xi}t\right)}d\xi \text{and } \widehat{\Lambda u}(\xi) = (1+|\xi|)^{-1}\hat{u}(\xi), \text{ for } \xi \in \mathbb{R} \tag{33}$$

Theorem. Fix $s<0$. There does not exist $aT>0$ such that admits a unique local solution defined on the interval $[-T,T]$ and such that for any fixed $t \in [-T,T]$ the map$\psi \longmapsto u(t)$

is C^2 differentiable at zero from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

The following lemma is found in equation 34.

$$\int_0^t S(t-\tau)\Lambda[(S(\tau)\psi)(S(\tau)\psi)_x]d\tau = c_0\int_{\mathbb{R}^2} e^{i(\xi x-p(\xi)t)}\frac{\xi}{1+|\xi|}\hat{\psi}(\eta)\hat{\psi}(\xi\eta)\frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}d\eta d\xi \tag{34}$$

where $p(\xi) = \frac{\xi}{1+|\xi|}$ and $\chi(\xi,\eta) = p(\eta)+p(\xi-\eta)-p(\xi)$.

Proof of Theorem 35. We define

$$\varphi(x,t) := \int_0^t S(t-\tau)\Lambda[(S(\tau)\psi)(S(\tau)\psi)_x]d\tau \tag{35}$$

Then, using the last lemma we have

$$\hat{\varphi}(\xi,t) = c_0\frac{\xi}{1+|\xi|}e^{-ip(\xi)t}\int_{\mathbb{R}}\hat{\psi}(\eta)\hat{\psi}(\xi-\eta)\frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}d\eta \tag{35}$$

In this case we consider

$$\hat{\psi}(\xi) := N^{-s}\chi_{[N,N+1]}(\xi), \quad \text{with } N\in\mathbb{N}, N \gg 1 \tag{36}$$

where χ_A denotes the characteristic function of A. Note that $\parallel \psi \parallel_{H^s(\mathbb{R})} \sim 1$ and using (3.7) we obtain

$$\hat{\varphi}(\xi,t) = c_0\frac{\xi}{1+|\xi|}e^{-p(\xi)t}N^{-2s}\int_{\Omega_\xi}\frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}d\eta \tag{37}$$

with $\Omega_\xi = \{\eta : \eta \in \operatorname{supp}\hat{\psi} \text{ and } \xi-\eta \in \operatorname{supp}\hat{\psi}\}$. Since $s<0$, we can choose $\epsilon>0$ such that $-s-\epsilon>0$. Now, consider $t = N^{-\epsilon}$ and note that for $\xi \in \left(2N+\frac{1}{2}, 2N+1\right)$ we have $\mu(\Omega_\xi) \gtrsim 1$. It is easy to see that

$$\chi(\xi,\eta) = \frac{\eta(\xi-\eta)(2+\xi)}{(1+\eta)(1+\xi-\eta)(1+\xi)} \leqslant 3, \quad \forall \eta, \xi-\eta \in [N,N+1] \tag{38}$$

Then, for N big enough we compute that

$$\| \varphi(\cdot,t) \|_{H^s(\mathbb{R})}^2 \gtrsim \int_{2N+\frac{1}{2}}^{2N+1} (1+|\xi|^2)^s N^{-4s} \frac{|\xi|^2}{(1+|\xi|)^2} |t|^2 \left| \int_{\Omega_\xi} \frac{e^{-it\chi(\xi,\eta)}-1}{t\chi(\xi,\eta)} d\eta \right|^2 d\xi$$
$$\gtrsim \int_{2N+\frac{1}{2}}^{2N+1} (1+|\xi|^2)^s N^{-4s} \frac{|\xi|^2}{(1+|\xi|)^2} |t|^2 \left| \int_{\Omega_\xi} \frac{\sin(t\chi(\xi,\eta))}{t\chi(\xi,\eta)} d\eta \right|^2 d\xi \quad (39)$$
$$\gtrsim N^{-4s} N^{2s} t^2.$$

Hence $1 \sim \| \psi \|_{H^s(\mathbb{R})} \gtrsim \| \varphi(\cdot,t) \|_{H^s(\mathbb{R})} \gtrsim N^{-s-\epsilon}$, which, for N≫1, presents a contradiction. Consequently, the demonstration in the nonperiodic scenario is thereby concluded.

4. Criteria for Stability in Equations of the BBM Type

The theoretical framework established for the rBO equation is broadened to encompass the family of equations in this section, wherein H is articulated as $\widehat{Hu}(n) = \alpha(n)\hat{u}(n), \quad \forall n \in \mathbb{Z}$

The It is posited that the symbol α represents an even, real, measurable, and locally bounded function defined on R, which adheres to the conditions outlined in (5.3). The traveling wave solutions ϕ_c in the equation (40) correspond to

$$cH\phi_c + (c-1)\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0 \quad (40)$$

Equation (41) is recognized as embodying the subsequent two principles of conservation.

$$E(u) = \frac{1}{2}\int_{-L}^{L} uHu - \frac{2}{(p+1)(p+2)} u^{p+2} dx \quad \text{and} \quad F(u) = \frac{1}{2}\int_{-L}^{L} uHu + u^2 dx (41)$$

and so By utilizing these principles, we deduce that the periodic solution is represented by Equation (42). ϕ_c satisfies $E'(\phi_c) + (c-1) \times F'(\phi_c) = 0$. Now, define

$$\mathcal{L} := E''(\phi_c) + (c-1)F''(\phi_c) = cH + (c-1) - \phi_c^p \quad (42)$$

Then the operator $\mathcal{L}: D(\mathcal{L}) \to L^2_{\text{per}}([-L,L])$is self-adjoint, closed, linear, and unbounded, and it is delineated on a dense subset of $L^2_{\text{per}}([-L,L])$. It is additionally apparent that Lϕ_c^'=0. The ensuing fundamental criteria emerge from the stability proof of the rBO equation

(C_0)A continuous spectrum of periodic solutions exhibiting significant characteristics is present for (43).

of the form $c \in I \subset \mathbb{R} \to \phi_c \in H^{m_2}_{\text{per}}([-L,L])(43)$;

(C_1) $\mathcal{L}$ has an unique negative eigenvalue and it is simple;

(C_2) the eigenvalue zero is simple;

$$\text{(C3)} \ \frac{d}{dc}\int_{-L}^{L} [\phi_c H\phi_c + \phi_c^2] dx > 0 \quad (44)$$

Subsequently, we delineate sufficient criteria for the operator L associated with problem to derive conditions (C_1) and (C_2). The ensuing statement represents the principal criterion for stability.

Let ϕ_c be a positive even solution of (44). Assume that $\hat{\phi}_c > 0$ and $\widehat{\phi_c^p} \in PF$ (2) discrete, then (C_1) and (C_2)) hold for the operator $\mathcal{L}$ in (45).

Evidence. It is essential to recognize that the operator L may be expressed as$\mathcal{L}u = (M + c)u - \phi_c^p u$, where $M = cH - 1$. The symbol of M is $\zeta(n) = c\alpha(n) - 1$. So, it is easy to see that for all $c \neq 0$ there exists $N_0 \in \mathbb{N}$ such that

$$B_1|n|^{m_1} \leqslant |\zeta(n)| \leqslant B_2(1 + |n|)^{m_2}, \quad \forall n \geqslant N_0 \tag{45}$$

where $B_1 = \frac{cA_1}{2}$ and $B_2 = cA_2 + 1$Subsequently, one may apply Theorem 5.1 to ascertain that, in the presence of the operator $\mathscr{L}$, the conditions $C1$ and $C2$ are satisfied. Applying the We obtain a continuous trajectory of positive cnoidal waves characterized by a period L through the application of the Implicit Function Theorem form:

$$c \in (c^*, +\infty) \mapsto \phi_c \in H_{\text{per}}^n([0, L]) \tag{46}$$

for all $n \in \mathbb{N}$, with $k := k(c)$ being a continuously differentiable function of c exhibiting a precise monotonic increase (figure 1). Subsequently, we select the velocity w of the solitary-wave solution ϕ_w such that it becomes $\psi_w$in (47) pertaining to the periodic traveling wave solution of the BBM equation.Specifically, for $c \in (c^*, +\infty)$, we define $w = w(c)$ as:

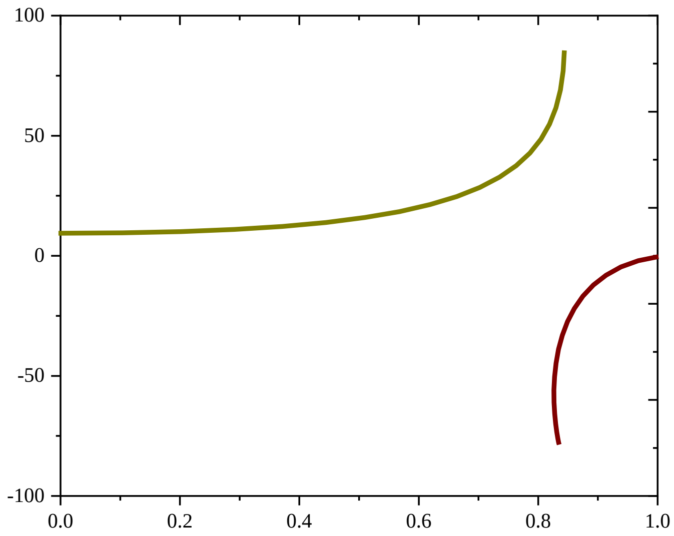

Figure. 1. Graphic of $c(k)$ with $L = 8$.

$$w(c) := \frac{16c\sqrt{k^4 - k^2 + 1}K'(k)}{16c\sqrt{k^4 - k^2 + 1}K'(k) - c + 1} \tag{47}$$

where $k = k(c) \in (0, k_L)$. Using the definition of w and equation (47), we find: $\sqrt{\frac{w}{w-1}} = \frac{LK'}{K}$

Then, from equations (48) and (49), we obtain the cnoidal profile:

$$\psi_{w(c)}(\xi) = \frac{12w}{L}\sqrt{\frac{w-1}{w}} + \frac{24K^2w}{L^2}\left[\text{dn}^2\left(\frac{2K\xi}{L}; k\right) - \frac{E}{K}\right] \tag{48}$$

Since $\frac{K(k)}{K'(k)} \in (0, L)$, for all $k \in (0, k_L)$, then for $c \in (c^*, +\infty)$ we obtain $w(c) \in (1, +\infty)$. Therefore, we get that the map

$$c \in (c^*, +\infty) \longmapsto \psi_{w(c)} \in H^n_{per}([0, L]) \tag{49}$$

is a uniformly symmetrical curve for every n in the set of natural numbers$n \in \mathbb{N}$.

The stability result for the BBM equation is stated as follows: Theorem. Assume $L > 2\pi$ is fixed. If $c > \frac{L^2}{L^2 - 4\pi^2}$, Subsequently, in accordance with the dynamics delineated by the BBM equation, the periodic traveling wave solution ϕ_c articulated in (50) exhibits stability.

Proof. From equation (50-53), we have:

$$\phi_c = a(k(c)) - \frac{24c}{L}\sqrt{\frac{w-1}{w}} + \frac{2c}{w}\psi_{w(k(c))} \tag{50}$$

where

$$a(k) = \frac{16cK}{L^2}[3E - (1 + k'^2)K] + c - 1 \tag{51}$$

$$\phi_c(x) = s(k(c)) + \frac{2c}{w}\psi_{w(k(c))}(x) \tag{52}$$

$$s\big(k(c)\big) := a\big(k(c)\big) - \frac{24c}{L}\sqrt{\frac{w-1}{w}} \tag{53}$$

We can then easily determine the Fourier coefficients of ϕ_c for $\in \mathbb{Z}$ and figure 2:

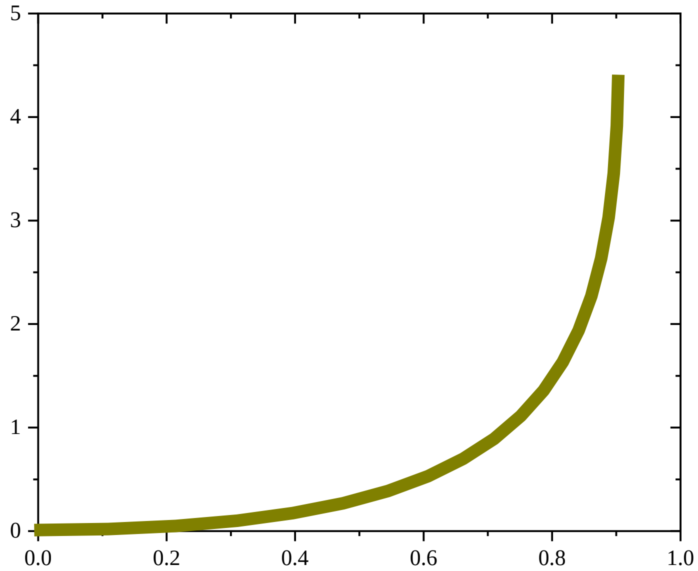

Figure. 2. Graphic of $\tilde{a}(k)$ with $L = 8$.

$$\hat{\phi}_c(n) = \begin{cases} a(k), & n = 0 \\ \frac{12c\pi}{L^2} n\operatorname{csch}\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right), & n \neq 0 \end{cases} \tag{54}$$

Now, by using that $\frac{c-1}{c} = \frac{16K^2\sqrt{1-k^2+k^4}}{L^2}$ we obtain

$$s(k) = c\left[\frac{16K^2}{L^2}\left(\sqrt{1-k^2+k^4} - 2 + k^2 + 3\frac{E}{K}\right) - \frac{24}{L^2}\frac{K(k)}{K(k')}\right] =: c\tilde{s}(k) \tag{55}$$

and

$$a(k) = \frac{16K^2c}{L^2}\left[3\frac{E}{K} - 2 + k^2 + \sqrt{1-k^2+k^4}\right] =: c\tilde{a}(k) \tag{56}$$

Considering that a(k) represents a positive and strictly monotonic increasing function within the interval (0,1), and the function s ˜ is also positive in $(0, k_L)$ (as $\tilde{a}$ is strictly increasing, Fig. 2), we can conclude that $\hat{\phi}_c \in$ PF(2) discrete.

Next, we demonstrate condition (C_3) in equation (57). Specifically, it is straightforward to verify that $\chi = -\frac{d}{dc}\phi_c$ satisfies $\mathcal{L}\chi = \phi_c - \phi_c''$. Then by Parseval theorem, it follows that $I = -\frac{L}{2}\frac{d}{dc}\left\|(1+|\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c\right\|_{l^2}^2$. But,

$$\begin{aligned} &\frac{d}{dc}\left\|(1+|\cdot|^2)^{\frac{1}{2}}\hat{\phi}_c\right\|_{l^2}^2 \\ &= 2a(k)\frac{da}{dk}\frac{dk}{dc} + c_1\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} (1+|n|^2)n^2\operatorname{csch}^2\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right) \\ &\quad + c_2((w-1)^3 w)^{-\frac{1}{2}}\frac{dw}{dk}\frac{dk}{dc}\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} (1+|n|^2)n^3\operatorname{csch}^2\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right)\coth\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right) \end{aligned} \tag{57}$$

where $c_1 = c_1(L,c) > 0$ and $c_2 = c_2(L,c) > 0$. To prove that $I < 0$ we only need to show that $\frac{dw}{dk} > 0$, because $b_n = (1+|n|^2)n^3\operatorname{csch}^2\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right)\coth\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right)$a sequence that is affirmative and $k = k(c)$ is a an algorithm that exhibits a strictly increasing behavior. Consequently, predicated on the established equivalence

$$\frac{dw}{dk} = \frac{2L^2K'K\left[k'\frac{dK}{dk} - K\frac{dK'}{dk}\right]}{(L^2K'^2 - K^2)^2}, \quad \forall k \in (0, k_0) \tag{58}$$

we have that $\frac{dw}{dk} > 0$ since $\frac{dK}{dk} > 0$ and $\frac{dK'}{dk} < 0$. Thusrefer to Benjamin for further details. The nonlinear stability of solitary waves associated with the rBO equation. Furthermore, proposed a periodic family of traveling wave solutions known as the stable positive cnoidal waves ϕ_c in $H^1_{\text{per}}([0,L])$ under the periodic flow of the BBM equation.

5. Conclusion

After regularization, the Benjamin-Ono and Benjamin-Bona-Mahony (BBM) equations, which are pivotal in describing wave phenomena like internal waves in deep stratified fluids and long surface waves in shallow waters, demonstrate improved mathematical properties and enhanced stability. Regularization addresses the issues of singularities and ill-posedness often encountered in their original formulations. By introducing corrective terms or smoothing mechanisms, regularization ensures that the solutions to these nonlinear partial differential equations are more tractable and less prone to abrupt changes or numerical instabilities. This adjustment not only refines the mathematical integrity of the models but also broadens their scope of applicability in accurately simulating real-world wave dynamics.

Furthermore, the regularized forms of the Benjamin-Ono and BBM equations retain their core physical relevance while being more suited for computational analysis. They offer a balanced approach by preserving key wave characteristics, such as amplitude and velocity profiles, over longer time frames without succumbing to the breakdowns typical in non-regularized models. This makes them invaluable for both theoretical research and practical applications, such as predicting the behavior of waves in oceans and other fluid bodies, where accuracy and stability are critical. Consequently, regularization enhances our understanding of complex wave interactions, making these equations more reliable tools in the study of fluid dynamics and related fields.

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